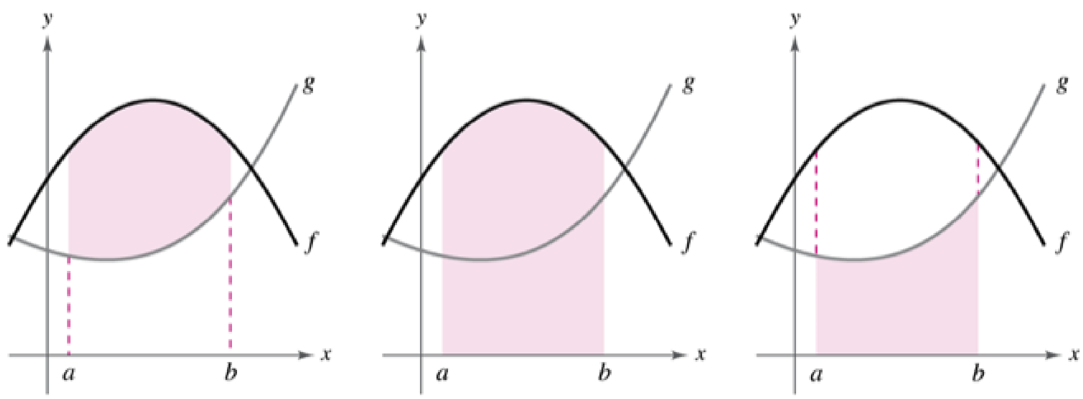
Area of a Region Between Two Curves



Area of region between f and g	=	Area of region under f	−	Area of region under g
$\int_a^b [f(x) - g(x)]\,dx$	=	$\int_a^b f(x)\,dx$	−	$\int_a^b g(x)\,dx$

If f and g are continuous on $\left[a,b\right]$ and $g(x) \le f(x)$ for all x in $\left[a,b\right]$, then the area of the region bounded by the graphs of f and g and the lines $x = a$ and $x = b$ is:

$$A = \int_a^b \left[f(x) - g(x)\right] dx$$

Example

Find the area of the region bounded by $y = x^2 + 2;\ y = -x;\ x = 0;\ x = 1$

Let $g(x) = -x$ and $f(x) = x^2 + 2$

$$A = \int_0^1 \left[\left(x^2 + 2\right) - \left(-x\right)\right] dx$$

$$= \left[\frac{x^3}{3} + \frac{x^2}{2} + 2x \right]_0^1$$

$$= \frac{1}{3} + \frac{1}{2} + 2 = \frac{17}{6} \approx 2.8333$$

Example

Find the area of the region between $f(x) = 2 - x^2$ and $g(x) = x$.

We need to find the intersection(s):

$$2 - x^2 = x$$

$$-x^2 - x + 2 = 0$$

$$-(x+2)(x-1) = 0$$

$$x = -2 \text{ or } x = 1$$

So,

$$A = \int_{-2}^{1} \left[\left(2 - x^2\right) - \left(x\right) \right] dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x \right]_{-2}^{1}$$

$$= \frac{9}{2}$$

Example

Find the area between sine and cosine.

$$\sin(x) = \cos(x)$$

$$\frac{\sin(x)}{\cos(x)} = 1$$

$$\tan(x) = 1$$

$$x = \frac{\pi}{4} \text{ or } x = \frac{5\pi}{4}$$

$$A = \int_{\pi/4}^{5\pi/4} \left[\sin(x) - \cos(x)\right] dx$$

$$= \left[-\cos(x) - \sin(x)\right]_{\pi/4}^{5\pi/4}$$

$$= 2\sqrt{2}$$

Example

Find the area of the region between $f(x) = x^3 - x^2 - 10x;\ g(x) = -x^2 + 2x$

$$3x^3 - x^2 - 10x = -x^2 + 2x$$

$$3x^3 - 12x = 0$$

$$3x(x-2)(x+2) = 0$$

$$x = -2, 0, 2$$

Find where the inequality changes using the graph or by plugging in:

$$g(x) \le f(x) \text{ on } [-2,0]$$

$$f(x) \le g(x) \text{ on } [0,2]$$

So,

$$A = \int_{-2}^{0} \left(3x^3 - 12x\right) dx + \int_{0}^{2} \left(-3x^3 + 12\right) dx$$

$$= \left[\frac{3x^4}{4} - 6x^2\right]_{-2}^{0} + \left[\frac{-3x^4}{4} + 6x^2\right]_{0}^{2}$$

$$= -(12 - 24) + (-12 + 24) = 24$$

Example-Horizontal Representative Rectangles

Find the area of the region bounded by $x = 3 - y^2$ and $x = y + 1$.

$$3 - y^2 = y + 1$$

$$0 = y^2 + y - 2$$

$$0 = (y + 2)(y - 1)$$

$$y = -2 \text{ and } y = 1$$

$f(y) = 3 - y^2$ and $g(y) = y + 1$

$$A = \int_{-2}^{1} \left[\left(3 - y^2 \right) - \left(y + 1 \right) \right] \, dy$$

$$A = \int_{-2}^{1} \left(y^2 - y + 2 \right) \, dy$$

$$= \left[-\frac{y^3}{3} - \frac{y^2}{2} + 2y \right]_{-2}^{1}$$

$$= \left(-\frac{1}{3} - \frac{1}{2} + 2 \right) - \left(\frac{8}{3} - 2 - 4 \right)$$

$$= \frac{9}{2}$$